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Mapping between hierarchical lattices by renormalisation and duality

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Abstract. The renormalisation group scheme of Migdal and Kadanoff is applied to a ferromagnetic Potts model on two hierarchical lattices: in the two cases the renormalisation transformation is split into two elementary processes (parallel bonds or series bonds) and each lattice can be mapped onto the other one. It is then easy to deduce the characteristic parameters (critical exponents, repulsive fixed points, etc) of one lattice from those of the other lattice. Moreover the classical geometrical duality offers a third possibility of mapping and the relationship between the renormalisation and the duality methods is presented in the peculiar case of a two-dimensional lattice.

1. Introduction

Recently Melrose [1] and Kaufman [2] have studied a class of hierarchical lattices in the framework of a general q-state Potts model. Hierarchical lattices have received much attention during recent years [3-10] because it is possible to obtain exact solutions by position space renormalisation group (PSRG) methods with such lattices. Otherwise, on Euclidean lattices, the models are not solved exactly and one must use approximate procedures, the best known solution being initially proposed by Migdal [11] and Kadanoff [12]; in a first step the initial lattice was transformed by bond moving in order to be able to apply the PSRG. Another method, called 'by decoration', was introduced by Emery and Swendsen [13]. Melrose [1] showed the existence of a duality relationship between the Migdal-Kadanoff (to be referred to as MK) and the Emery-Swendsen (ES) approaches.

The aim of the present paper is to generalise and to make precise the results obtained by Melrose [1] and Kaufman [2]; in particular an attempt will be made to characterise and define the notions of 'mapping' and 'duality' with two special hierarchical lattices: a rigorous relationship between these two sorts of transformation will be given.

A hierarchical lattice can be defined such that starting from one bond between two sites, this bond can be replaced, in a first step, by a given pattern. Then this transformation is repeated iteratively giving a dilation symmetry structure: figure 1 shows the best known hierarchical structure, the diamond lattice, for the first two iterative steps. The two peculiar hierarchical lattices studied here are defined as follows.

(a) The first one is represented in figure 2(a). It is a generalisation of that studied by Kaufman [2] because we consider a chain of s links, each one made up by p parallel

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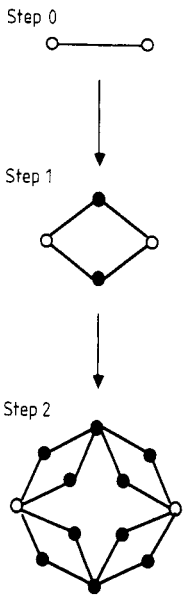

Figure 1. Scheme for the iterative construction of the diamond hierarchical lattice.

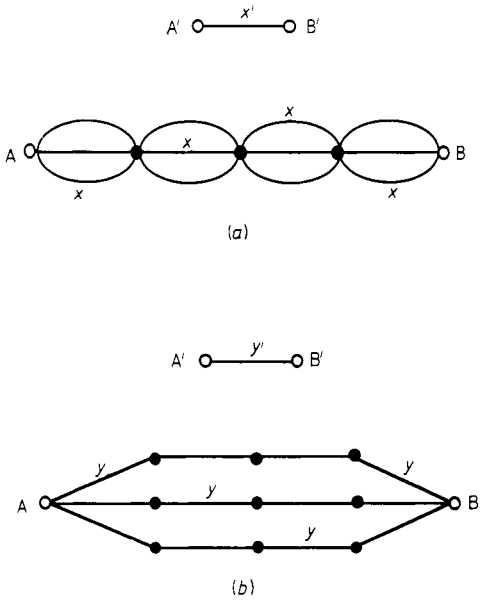

Figure 2. Patterns of the two hierarchical lattices in the case $p=3$, $s=4$. (a) Necklace lattice. (b) Diamond lattice.

bonds (in [2] $p=s$, corresponding to a two-dimensional system). It will be called the 'necklace' lattice.

(b) The second hierarchical lattice is a generalisation of the diamond one (figure 2(b)) and will be called improperly, but simply, the 'diamond' lattice. The basic pattern contains p lines of s bonds.

Two mappings will be defined on renormalisation considerations in order to deduce the physical results corresponding to one lattice from the results given by the other lattice. It will be seen that one of these mappings is very simple, the other one being much more complicated. These results can soon be obtained from the simple generic Ising case ($q = 2$) [14].

The notion of geometrical duality is then introduced in the peculiar case $s = p$: it is then possible to define a third mapping. The combination of the duality procedure with the simple mapping obtained by renormalisation simply gives the sophisticated mapping.

2. The model

The notation of the Potts model [15, 16] is briefly recalled. Each site interacts with its nearest neighbours with a strength J ($J > 0$ for a ferromagnetic interaction, $J < 0$ for an antiferromagnetic one). For a given configuration of the system the Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta S_i S_j \tag{2.1}$$

where the eigenvalue S_i, corresponding to the site labelled i, is included in the set $[1, \ldots, q]$, δ is the usual Kronecker symbol and the bracket $\langle\ \rangle$ signifies that only first-neighbour sites are taken into consideration. Then the partition function is given by

$$Z = \sum_{\{S\}} \exp[-(\mathcal{H}/kT)] \tag{2.2}$$

where $\{S\}$ stands for a sum over all the configurations of the system. For convenience the parameter $\exp(J/kT)$, which is equivalent to the temperature T, will be designated by x (y) for the necklace (diamond) lattice. Figure 2 presents one step of the iterative building for these two hierarchical lattices in a general case, i.e. for a non-integer dimensionality d. Here $s = 4$ and $p = 3$ and then $d = 1 + \ln 3/\ln 4$. The two first initial sites for the two lattices are called A′ and B′, and become A and B after the first iterative step. The parameter p counts the number of bonds starting from (arriving on) the points A(B); the parameter s measures the distance between A and B in bond units.

The procedure of elaborating one lattice step by step from an original link is opposite to the classical renormalisation method which aims to reduce the number of degrees of freedom. The application of a renormalisation step will be noted as by Itzykson and Luck (IL) [17]

$$(a)\ \text{necklace:}\quad x' = T_{\mathrm{N}}(x) \tag{2.3}$$

$$(b)\ \text{diamond:}\quad y' = T_{\mathrm{D}}(y). \tag{2.4}$$

The transformations T_{N} and T_{D} are not independent and they can be expressed in terms of elementary processes which will be described in detail in the next section. Here an important remark should be made: in fact, the necklace lattice reproduces a possibility of bond moving according to the MK method of PSRG applied to Euclidean lattices.

3. Renormalisation of the necklace and the diamond lattices: two ways for mapping

These two hierarchical lattices can be built from two elementary mechanisms and, in a first intermediary step, one can replace the existing bond (figure 3) by

(a) p parallel bonds

(b) s series bonds.

If operation (a) is done first, followed by operation (b) this succession being noted (a-b), the necklace lattice is obtained. Evidently the succession (b-a) gives the diamond lattice. The two renormalisation transformations of figure 3 can be written analytically (see, for example, [17]):

$$\text{(a)}\quad P_p : x \to x^{(1)} = P_p(x) = x^p \tag{3.1}$$

$$\text{(b)}\quad S_s : y \to y^{(1)} = S_s(y) = \frac{(y+q-1)^s + (q-1)(y-1)^s}{(y+q-1)^s - (y-1)^s}. \tag{3.2}$$

In order to simplify the notation and avoid ambiguity P_p and S_s will be denoted P and S, respectively. The functions T_N and T_D given by the expressions (2.3) and (2.4) can be written as compositions of the functions P and S:

$$x' = T_N(x) = S(x^{(1)}) = S[P(x)] \tag{3.3}$$

$$y' = T_D(y) = P(y^{(1)}) = P[S(y)] \tag{3.4}$$

and symbolically one can write

$$T_N = S \circ P \tag{3.5}$$

$$T_D = P \circ S. \tag{3.6}$$

Now let us study some properties of P and S. If I is the inversion operator, i.e. the operator transforming an attractive (repulsive) bond into a repulsive (attractive) one, the Potts state of one site limiting the considered bond is changed:

$$x \to x^{-1}. \tag{3.7}$$

It is easy to see from (3.1) that

$$P \circ I = I \circ P \qquad \text{for every } q.$$

Then two inverse points, x and x^{-1}, of (3.7) are kept inverse by the transformation P and it induces no frustration. This result is not difficult to understand because only parallel bonds are considered: if the Potts state of one limiting site is changed, one must take the opposite sign for every bond energy. The problem is different for the series gathering, where in general a frustration appears which corresponds to the exotic

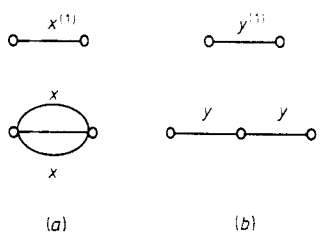

Figure 3. Elementary steps of the renormalisation transformation. (*a*) Parallel bonds. (*b*) Series bonds.

phases first introduced by Berker [18]. In the peculiar case $q = 2$ one can show easily from (3.2) that

$$S \circ I = S \qquad \text{for } s \text{ even}$$

$$S \circ I = I \circ S \qquad \text{for } s \text{ odd.}$$

Two inverse points, y and y^{-1}, have the same image and the frustration has been avoided.

Moreover P is a good mapping of the variable x into the variable y, i.e. if one supposes.

$$y = P(x) \tag{3.8}$$

then from (3.4)

$$y' = P[S\{P(x)\}]$$

and with (3.3)

$$y' = P(x'). \tag{3.9}$$

The expressions (3.8) and (3.9) prove the stated property. A similar possibility of mapping for the variable y into the variable x with the aid of the transformation S is demonstrated with the same method:

$$x = S(y) \rightarrow x' = S(y'). \tag{3.10}$$

It is possible to summarise these two mappings with a diagram (see figure 4) where each line is horizontally shifted from the preceding one by half a step.

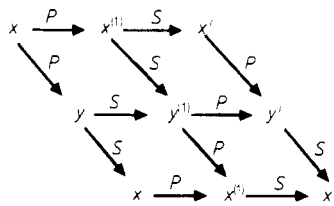

Figure 4. Diagram presenting the renormalisation steps (horizontal arrows) for the necklace (variable x) and the diamond (variable y) lattices. Sloped arrows are associated to the mapping from one lattice upon the other.

Then, with these two mappings, all the parameters corresponding to one of the two hierarchical lattices can be deduced from those of the other lattice. It should be noted that the transformation P, such that $y = P(x) = x^p$, offers a mapping which is much simpler than that for the transformation S. In particular for the unstable fixed points (characterised by an asterisk):

$$y^* = x^{*p}. \tag{3.11}$$

From this property it is easy to deduce the Julia set [19] of the diamond lattice from the Julia set of the necklace: if (r, θ) are the coordinates of a given point included into the Julia set of the necklace, the corresponding point for the diamond lattice is $(r^p, p\theta)$. Figure 5 presents the Julia sets of the necklace (a) and (b) and the diamond (c) and (d) with $q = 2$, in the cases $(p = 4, s = 2)$ and $(p = 2, s = 4)$.

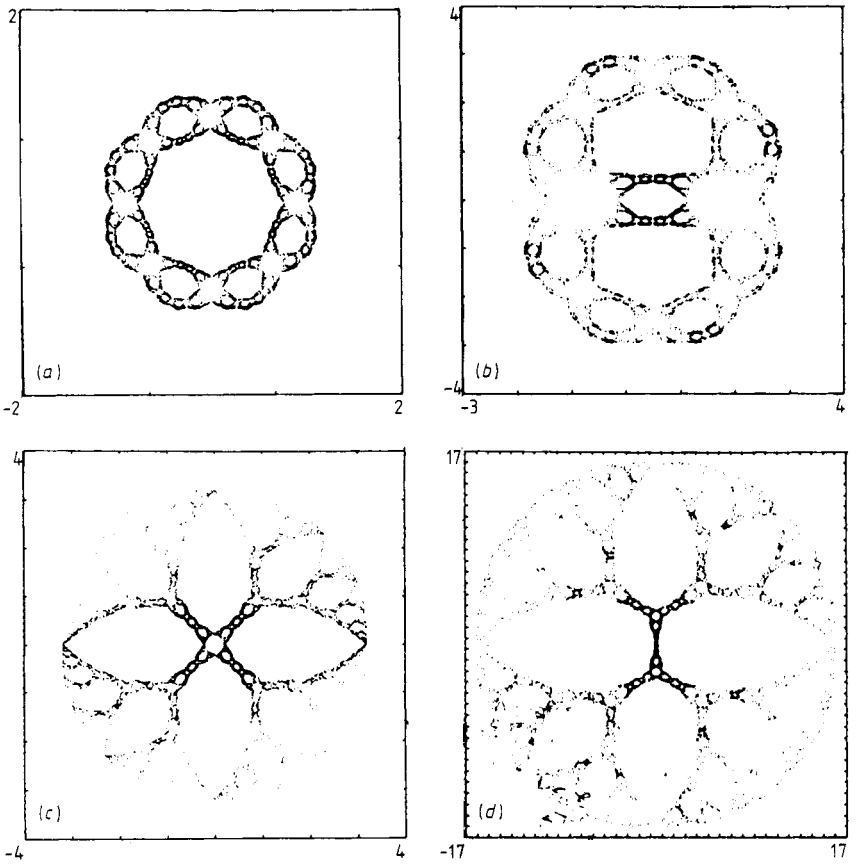

Figure 5. Julia set of the necklace (*a*), (*b*) and the diamond (*c*), (*d*) lattices obtained for $q=2$. The horizontal (vertical) axis corresponds to the real (imaginary) part of the roots obtained from the iterative transformations. The geometrical parameters are: necklace: (*a*) $p=4$ and $s=2$; (*b*) $p=2$ and $s=4$; diamond: (*c*) $p=4$ and $s=2$; (*d*) $p=2$ and $s=4$.

It is also interesting to look at the behaviour of the transformations S and P at the continuous limit of rescaling studied by IL [17]; the dimensionality d depends on the space dilation parameter $1+\varepsilon$ and the relations between s, p and d are

$$s=1+\varepsilon \qquad \text{where } \varepsilon \ll 1 \tag{3.12a}$$

$$p=1+(d-1)\varepsilon \tag{3.12b}$$

and the two transformations S and P are given by, at first order in ε and the variable z standing for x or y,

$$P(z)=z+\varepsilon\Pi(z) \tag{3.13a}$$

$$S(z)=z+\varepsilon\sigma(z) \tag{3.13b}$$

where

$$\Pi(z)=(d-1)z\ln z \tag{3.14a}$$

$$\sigma(z)=\frac{(z+q-1)(z-1)}{q}\ln\frac{z-1}{z+q-1} \tag{3.14b}$$

and

$$x' = T_N(x) = x + \varepsilon\beta(x) \tag{3.15a}$$

$$y' = T_D(y) = y + \varepsilon\beta(y) \tag{3.15b}$$

with

$$\beta(z) = \Pi(z) + \sigma(z). \tag{3.15c}$$

The two transformations T_N and T_D become identical in the continuous limit.

4. Duality: a third way for mapping

There is another method permitting the transformation of the necklace into the diamond lattice (or vice versa): this is the well known geometrical duality as initially defined by Kramers and Wannier [20]. Figure 6 presents for a necklace (a) and a diamond (b) initial lattices (dots and full lines) the corresponding dual lattices (crosses and broken lines). The dual of a necklace is a diamond (and vice versa) and the parameters s and p are exchanged by the transformation from the initial lattice into its dual. With the notation of [17], the parameter associated to a bond is $u = \delta(x)$ for the necklace and $v = \delta(y)$ for the diamond, where the duality transformation δ obeys

$$(1-x)(1-\delta(x)) = q \tag{4.1}$$

the variable y giving the same expression. Moreover δ is evidently involutive:

$$\delta \circ \delta = \mathbb{1}.$$

In the important particular case $s = p = b(d = 2)$ the dual of figure 6(a) is identical to the original lattice of figure 6(b) and vice versa. Then $u = \delta(x) = y$ and $v = \delta(y) = x$ and the duality mapping can be written

$$(1-x)(1-y) = q. \tag{4.2}$$

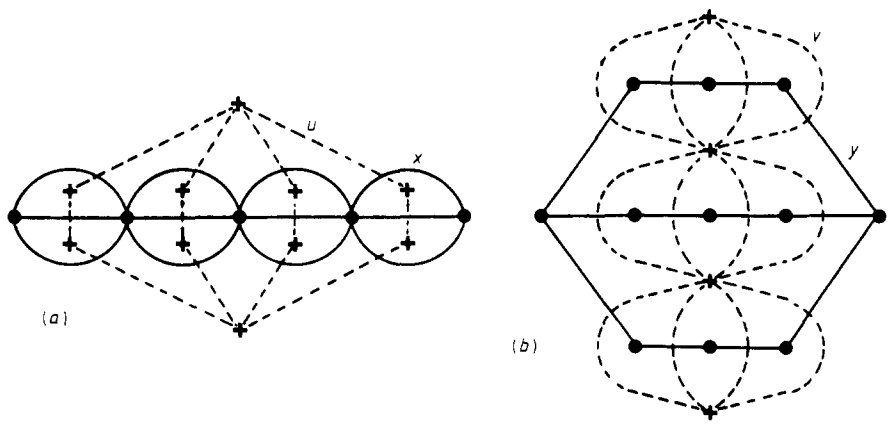

Figure 6. Geometrical duality in a general case. Here, in the initial lattice represented by dots and full lines the geometrical parameters are $p_i = 4$, $s_i = 3$. In the dual lattice (broken lines and crosses) the geometrical parameters are inverted: $p_d = 4$, $s_d = 3$. (a) Necklace. (b) Diamond.

This gives, taking (3.8) into account, a closed expression to calculate the unstable fixed points for the two lattices:

$$(1-x^*)(1-x^{*b}) = q \tag{4.3a}$$

$$(1-y^{*1/b})(1-y^*) = q. \tag{4.3b}$$

In the limit $b \to 1$, $x^* \to y^*$ the two lattices are self-dual; then the expressions (4.3) reduce to

$$x^* = y^* = 1+\sqrt{q} \tag{4.4}$$

and the corresponding critical exponent, classically denoted by ν, is

$$\nu = [2-\sqrt{q}\ln(1+\sqrt{q})]^{-1}. \tag{4.5}$$

Tables 1 and 2 present critical parameters and exponents for the necklace (which corresponds to a Euclidean lattice after the MK bond moving) and the diamond lattices, with different q (table 1 where b is fixed and equal to 2) and b (table 2 in the Ising case $q=2$) parameters.

Always in the case $d=2$ it is now possible to generalise the results of IL [17]: starting, for instance, from a pattern of necklaces and remembering that $x=\delta(y)$ and $y=\delta(x)$ we can transform it by duality as illustrated in figure 7. The transformation $x \to y^{(1)}$ in figure 7 is exactly what is called the 'general duality transformation' in [17]

Table 1. Critical constants for the two lattices (the triangular lattice is renormalised by the MK bond moving method and the critical parameters are the same as for the necklace lattice) with $s=p=2$ and varying q. The 'exact' values for ν are those given by den Nijs [21].

Q	y^* (diamond)	x^*(MK)	ν^*	x_c (triangle exact)	ν (exact)
1	2.618 034	1.618 034 $(1+\sqrt{5})/2$	1.635 280	1.532 089	$\frac{4}{5}$
2	3.382 976	1.839 287	1.338 266	1.732 051 ($\sqrt{3}$)	1
3	4	2	1.204 710	1.879 385	$\frac{5}{6}$
4	4.538 5847	2.130 395	1.124 161	2	$\frac{2}{3}$
100	25.620 448	5.061 6645		4.827 183	

Table 2. Critical constants for the necklace (≡to the triangular lattice constants) and the diamond lattices for a given q (here this is only the simplest Ising model $q=2$ which is taken under consideration) and varying s and p parameters (always with $s=p$ in order that $d=2$).

b	y^*	x^*	ν^* (approximate)
1	$1+\sqrt{2}$	$1+\sqrt{2}$	1.3271
2	3.382 976	1.839 287	1.3383
3	4.236 068 $2+\sqrt{5}$	1.618 034 $(1+\sqrt{5})/2$	1.3548
4	5.023 384	1.497 094	1.3706

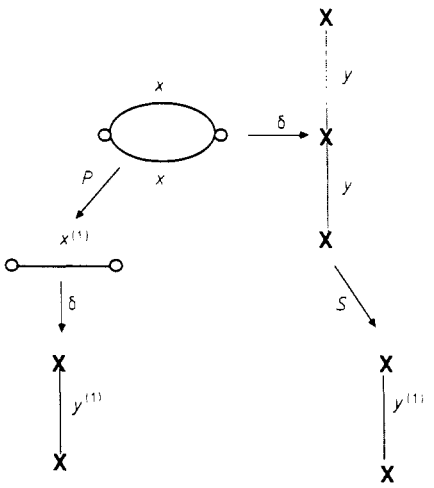

Figure 7. Diagram showing the combination of renormalisation (S or P) and duality (δ) transformations in order to obtain the general duality transformations: (a) D_N and (b) D_D. The sites of the initial lattices are represented by circles, the bonds by full lines. For the dual lattices the crosses correspond to the sites and the dotted lines to the bonds.

and denoted by D. This transformation will be denoted $D_N(D_D)$ for the necklace (diamond) lattice and

$$D_N = S \circ \delta = \delta \circ P \tag{4.6a}$$

$$D_D = P \circ \delta = \delta \circ S. \tag{4.6b}$$

The case $P \circ \delta$ corresponds to the general duality transformation $D(x)$ given by formula (5.14) in [17]. The different possible steps of the transformations are given in figure 7.

The iteration of the duality transformations leads to

$$D_N \circ D_N = (S \circ \delta) \circ (\delta \circ P) = S \circ P = T_N \tag{4.7a}$$

$$D_D \circ D_D = (P \circ \delta) \circ (\delta \circ S) = P \circ S = T_D. \tag{4.7b}$$

The relations between D_N and D_D, on the one hand, and between T_N and T_D, on the other hand, are ($\delta = \delta^{-1}$, δ being involutive):

$$D_N = \delta \circ D_D \circ \delta \qquad T_N = \delta \circ T_D \circ \delta \tag{4.8a}$$

$$D_D = \delta \circ D_N \circ \delta \qquad T_D = \delta \circ T_N \circ \delta. \tag{4.8b}$$

All these expressions are summarised in the commutative diagram of figure 8 where only the transformation δ is reversible.

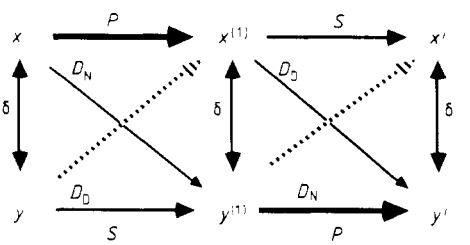

Figure 8. Summary of the different possible transformations connecting the necklace and the diamond lattices.

Finally it is interesting to note that the difficult mapping P introduced in § 3 can be easily deduced from the simplest one (S) and the duality transformation δ (always in the case $s = p$)

$$P = \delta \circ S \circ \delta. \tag{4.9}$$

5. Conclusion

It has been shown [1-3] that the PSRG method is exact only on hierarchical (self-similar) lattices. On Euclidean lattices one must develop approximate methods for bond moving. In this paper, we show that the necklace lattice is the hierarchical lattice for which the MK transformation is exact. The necklace can be mapped on the well known diamond (generalised) hierarchical lattice by two methods.

(a) The geometrical duality, but only in the peculiar case $s = p$; otherwise the parameters s and p are exchanged by the duality transformation. Moreover one must note that it is only in the continuum limit that the necklace and the diamond lattices are self-dual.

(b) The classical renormalisation method which permits the necklace to be mapped on the diamond and vice versa.

(c) Only two relations are necessary for the two mappings (necklace $\rightleftarrows$ diamond): when the geometrical duality works (d is an integer) the duality expression (4.1) and the simplest mapping relation (3.1) permit the derivation of the other mapping expression by renormalisation (3.2). Here the general case is taken under consideration but, in general, the studied systems have integer dimensionalities.

Consequently all results concerning Euclidean lattices studied by a bond moving PSRG method can be obtained through the medium of the mapping methods and the duality procedure (in the limiting case $s = p$) mentioned above: the two classes of problems are perfectly equivalent.

Acknowledgments

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